Title: What You See Is Not Always What You Get!

Brief Overview:

This unit will examine the relationship between the horizontal distance between an object and its reflected image in water vs. height between the water level and the eye of the observer.

NCTM 2000 Principles for School Mathematics:

- **Equity:** *Excellence in mathematics education requires equity - high expectations and strong support for all students.*

- **Curriculum:** *A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades.*

- **Teaching:** *Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well.*

- **Learning:** *Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.*

- **Assessment:** *Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.*

- **Technology:** *Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.*

Links to NCTM 2000 Standards:

- **Content Standards**

  Algebra
  Students will see patterns in their measurements and will construct an algebraic model from the data they collect.

  Geometry
  Students will draw geometric diagrams to represent quantities in their data, and use these diagrams to visualize relationships between quantities.

  Measurement
  Students will collect experimental data by making measurements. In addition, they will understand the necessity to make measurements in a consistent manner.

<u>Data Analysis and Probability</u>
Students will collect and analyze data, use the data to construct a mathematical model, and use the model to solve problems and make predictions.

- **Process Standards**

<u>Mathematics as Problem Solving, Reasoning and Proof, Communication, Connections, and Representation</u>
These five process standards are threads that integrate throughout the unit, although they may not be specifically addressed in the unit. They emphasize the need to help students develop the process that are the major means for doing mathematics, thinking about mathematics, understanding mathematics, and communicating mathematics.

Links to Maryland High School Mathematics Core Learning Units:

Functions and Algebra

- **1.1.1**
  Students will describe and extend patterns, and functional relationships that are expressed numerically, algebraically, and geometrically.

- **1.1.2**
  Students will represent patterns and functional relationships in a table, as a graph, and/or by mathematical expression.

- **1.2.4**
  Students will describe how the graphical model of a non-linear function represents a given problem and will estimate the solution.

Geometry, Measurement, and Reasoning

- **2.1.2**
  Students will identify and verify properties of the geometric figures using concepts from algebra and using the coordinate plane.

- **2.2.2**
  Students will solve problems using two-dimensional figures and/or right triangle trigonometry.

- **2.3.1**
  Students will use algebraic and geometric properties to measure indirectly.

Data Analysis and Probability

- **3.1.1**
  Students will design and/or conduct an investigation that uses statistical methods to analyze data and communicate results.

- **3.2.2**
  Students will make predictions by finding and using a line of best fit and by using a given curve of best fit.

Grade/Level:

Grades 8-12; Algebra I, Geometry, Algebra II, Trigonometry, Pre-Calculus

Duration/Length:

Two - 90 minute blocks

Prerequisite Knowledge:

Students should have working knowledge of the following skills:

- Plotting coordinates
- Using scatter plots

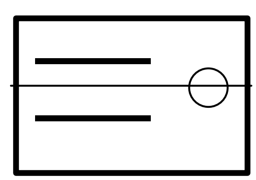
Student Outcomes:

Students will be able to:

- collect and organize data.
- examine data in order to describe a relationship between two variables.
- select an appropriate form of a mathematical model.
- use the calculator to fit a curve to collected data.
- use a model to interpolate values.

Materials/Resources/Printed Materials:

- A relatively long and deep transparent container
- A ruler or long piece of wire
- A small bright object such as a penny, dime, straight pin, etc
- Graphing calculator
- A yardstick
- Three tape measures or three pieces of masking tape approximately the length of the container. Start at the end of each piece of tape and mark off in convenient units of length, quarter inches or centimeters.
- A 3” x 5” index card with a hole punched near one edge. Cut two horizontal parallel slits approximately ½” longer than the width of the yard stick opposite the hole. Draw a horizontal line from the center of the hole between the two parallel slits.



- A tripod with clamp (optional)

- Water to fill the container
- Copy of experiment directions (below in this packet)
- Lab Write Up worksheets

Development/Procedures:

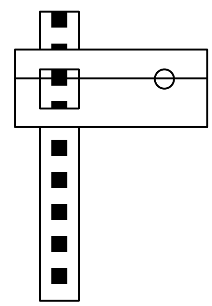
Set Up

Step 1: Tape the two tape measures along the upper edges of the two longest sides of the container.

Step 2: Fill the container with water. You want the water level to be within a quarter inch of the top.

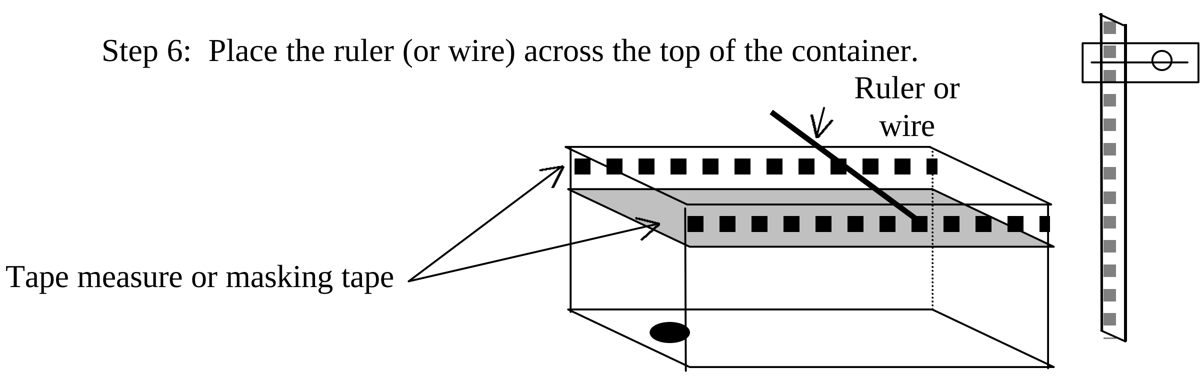
Step 3: Place the small object in the center of the far end of the container.

Step 4: Attach the index card to the edge of the yard stick by sliding the yard stick through the two horizontal slits. Make sure that the hole is not blocked.



Step 5: Place the tripod on the table near the side of the container that is opposite the small object. Attach the yard stick to the tripod using the clamp, have someone hold the yardstick on the table or attach it to the side of the container using masking tape.

Step 6: Place the ruler (or wire) across the top of the container.



Experiment Directions

Measure the following lengths:

- **Water depth**. This length is denoted by the letter **a** in the diagram below.
- **Distance of Object** - Horizontal distance from the small object to the index card (eye of the observer). This distance is denoted by the letter **b** in the diagram below.

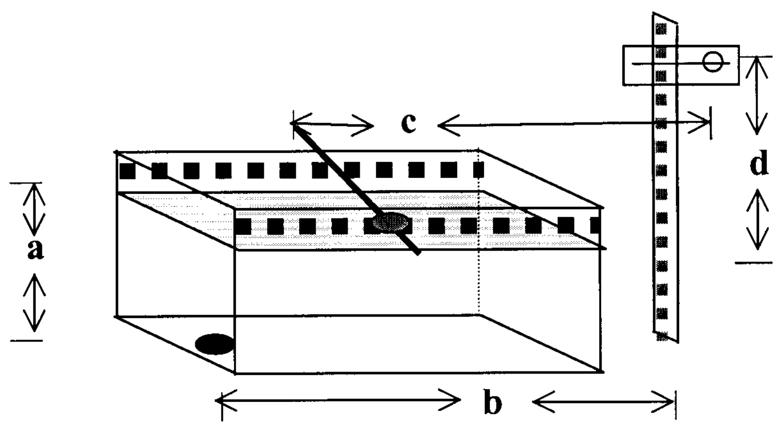
Determine the measures of the following:

- **Distance of Image** - Horizontal distance from the reflected image to the index card (eye of the observer) :
  Choose one of your teammates to be the observer.
  The observer will look through the pencil hole in the index card sighting the object at the opposite end of the container. He or she will then slide the index card down the yardstick as far as possible, making sure that the small object is still clearly visible and that the pencil hole is above water level.

  Choose two other teammates to be the readers.
  One reader will slide the ruler (or wire) along the top of the container and away from the observer. The observer will tell the reader to stop when the ruler (or wire) appears to be on top and in the center of the small object. (See teacher's notes for more details.)

  This distance is denoted by the letter **c** in the diagram.

- **Height Above Water Level** - Distance from the top of the water to the eye of the observer. This distance is denoted by the letter **d** in the diagram below



Increase the **Height Above Water Level** by sliding the index card up the yardstick by a small increment, one to two inches or two to four centimeters. Record the new values for the **Height Above Water** and the **Distance of the image**. Repeat this process to obtain at least 12 data points.

Assessment:

Student achievement will be based on the student lab write-up sheet.

Extension/Follow Up:

- Algebraically, find an equation that “should” model your data by drawing a diagram of the problem situation, applying Snell’s Law and right triangle trigonometry.
- Try the experiment using a clear cooking oil. Use your data and Snell’s Law to determine the index of refraction.

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LAB WRITE UP

Name: ______________________________

WHAT YOU SEE IS NOT ALWAYS WHAT YOU GET!

Date: ________________________________

Problem: The carnival has been in town several days. Every day after they have finished their chores, Willie the Pooch and Tiger make a bee-line to carnival. They are determined to win at this new game called "What You See Is Not Always What You Get." The BIG prize, if you are successful, is a gallon of honey.

They have now attempted the game five times. Each time taking careful measures to make sure that they sight the ball that is submerged in water through the peek hole and that they precisely align the arrow directly over the point where they "think" the ball is located before dropping the arrow. But each time they were unsuccessful in hitting the ball.

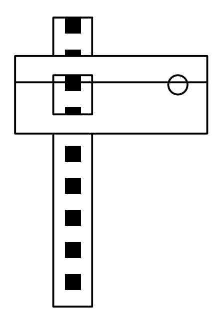
What advice can you and teammates give to Pooch and Tiger to guarantee that they will be successful at their next attempt?

<u>Experiment Set Up</u>

Step 1: Tape the two tape measures along the upper edges of the two longest sides of the container.

Step 2: Fill the container with water. You want the water level to be within a quarter inch of the top.

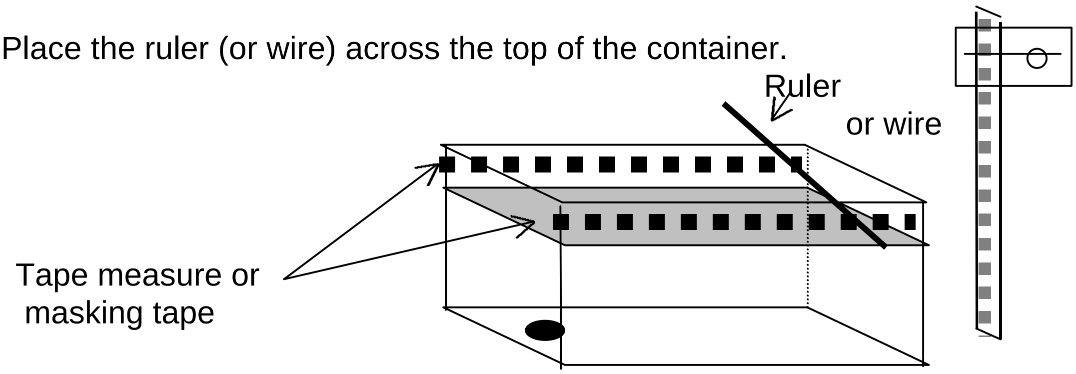
Step 3: Place the small object in the center near the far end of the bottom of the container.



Step 4: Attach the index card to the edge of the yard stick by sliding the yard stick through the two horizontal slits. Make sure that the hole is not blocked.

Step 5: Place the tripod on the table near the side of the container that is opposite the small object. Attach the yardstick to the tripod using the clamp, have someone hold the yardstick on the table or attach it to the side of the container using masking tape.

Step 6: Place the ruler (or wire) across the top of the container.



Experiment Directions

Measure the following lengths:

- **Water depth**. This length is denoted by the letter **a** in the diagram below.
- **Distance of Object** - Horizontal distance from the small object to the index card (eye of the observer). This distance is denoted by the letter **b** in the diagram below.

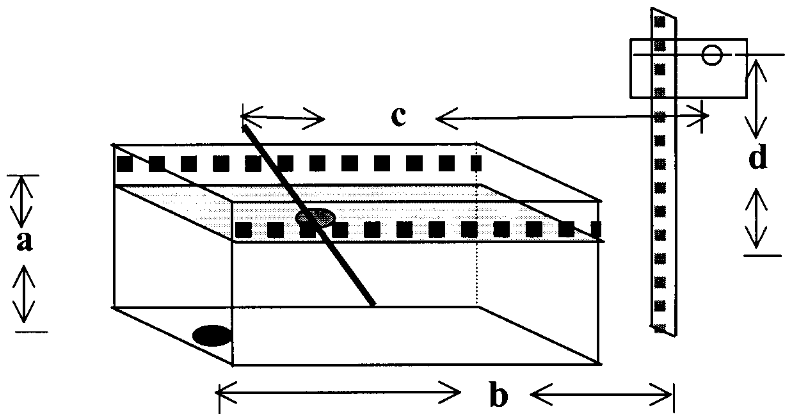
Determine the measures of the following:

- **Distance of Image** - Horizontal distance from the reflected image to the index card (eye of the observer):
  Choose one of your teammates to be the observer. The observer will look through the pencil hole in the index card sighting the object at the opposite end of the container. He or she will then slide the index card down the yardstick as far as possible, making sure that the small object is still clearly visible and that the pencil hole is above water level.

  Choose two other teammates to be the readers. One reader will slide the ruler (or wire) along the top of the container and away from the observer. The observer will tell the reader to stop when the ruler (or wire) appears to be on top and in the center of the small object.

  This distance is denoted by the letter **c** in the diagram.

- **Height Above Water Level** - Distance from the top of the water to the eye of the observer. This distance is denoted by the letter **d** in the diagram below.



Increase the **Height Above Water Level** by sliding the index card up the yardstick by a small increment; one to two inches or two to four centimeters. Determine the **Distance of the image**. Record the new values for the **Height above water** and the **Distance of the image**. Repeat this process to obtain at least 12 data points.

<u>Answer Questions 1 & 2 before starting the experiment</u>

1. What is the purpose of the experiment? (Why are we doing this?)

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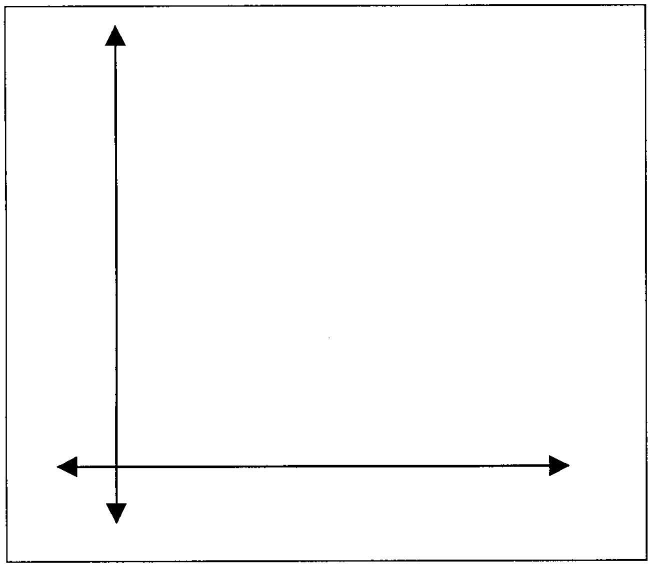
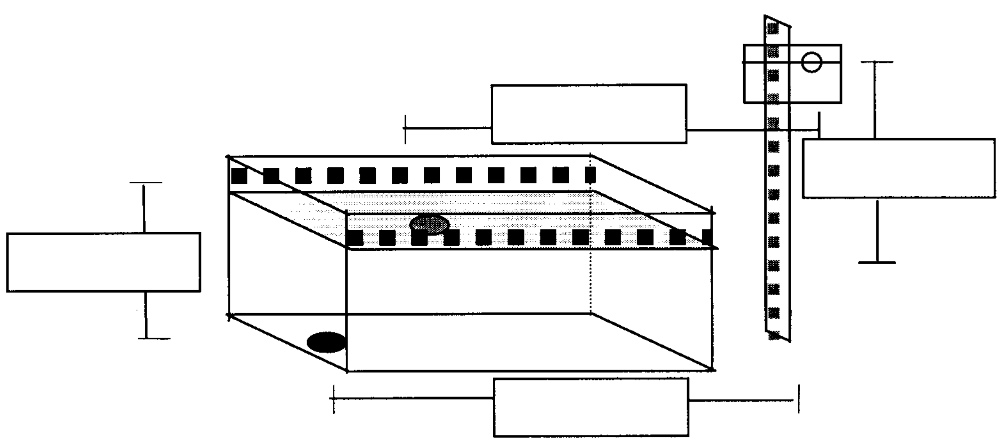
2. Before you collect your data, make a conjecture about this experiment. What do you think the graph of your data would look like? Be sure to label your axes and explain your reasoning.



How can we find the horizontal distance between the object and the reflected image?

3. Data Collection

   Run the experiment following the directions above. Record your experiment data in the space provided:



Height Above Water	Distance of image				Horizontal Distance Between the object and image
	Trial 1	Trial 2	Trial 3	Average or Median	

4. Graph:

a. The independent variable represents ______________________________ .

The dependent variable represents _____________________________. Units______

b. List the elements of the domain:

List the elements of the range:

TI-83 Instructions:

Press $\boxed{\text{STAT}}$, EDIT (or #1) and enter the values from your “Height above water “ column above into L1. Enter the “Horizontal distance between the object and its image” from your table into L2. To view the graph of your data, press $\boxed{\text{2nd}}$ $\boxed{\text{Y=}}$ to obtain **STAT PLOT** and select **1:Plot 1**. Select ON. Select the scatter plot (first icon). In Xlist, enter L1 and in Ylist, enter L2. Select your favorite mark. Press $\boxed{\text{ZOOM}}$ select **9: ZoomStat**.

5. How does this graph compare to the graph you drew in Question #2?

Please copy the graph from your calculator onto the graph below. Be sure to label your axes with your independent and dependent variables. Indicate the scale for each axis.

6. Data Analysis:

a. Describe what you see on your graph. Is the data linear?

b. What do your data and graph tell you about the physical situation?

c. Does it make sense to connect your points with a smooth curve? ☐ yes ☐ no
Explain.

____

d. Does it make sense to extend the curve beyond your data points? ☐ yes ☐ no
Explain.

____

e. Draw a smooth curve through the data points. State the new domain and range.

Domain ______________________________ Range________________________

f. Is there a x-intercept? ☐ yes ☐ no . If yes, explain its relationship to the physical
situation. If no, explain why not, in terms of the physical situation.

____

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____

g. Is there a y-intercept? ☐ yes ☐ no . If yes, explain its relationship to the physical situation. If no, explain why not, in terms of the physical situation.

____

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____

____

7. What type of function might the curve represent? ______________________________

8. Select at least one inch (or 2 centimeters) greater than your maximum horizontal distance between the object and its image. Complete the table below using your curve to predict the corresponding height above water level.

Horizontal Distance Between Object and Image	Height Above Water Level

9. Choose a value between your maximum and minimum heights above water level that is <u>not</u> in your table. Complete the table below using your curve to predict the value for the corresponding horizontal distance between the object and image.

Height Above Water Level	Horizontal Distance Between Object and Image

10. Pooch and Tiger will succeed for the first time because of your input. What advice did you and your teammate give them?

____

____

____

11. If Pooch is 10.5 inches (or 26.5 cm)above water level and sights the image of the ball approximately 7 inches away, based on your data, at what point should he place the arrow? Show all work in the space provided.

****Extension for more advanced classes *****

12. Consider all of the families of curves that you are familiar with. Choose a family of curves and try to find an equation that closely fits your data. You may use your calculator.

What family of curves did you choose?_________________________________

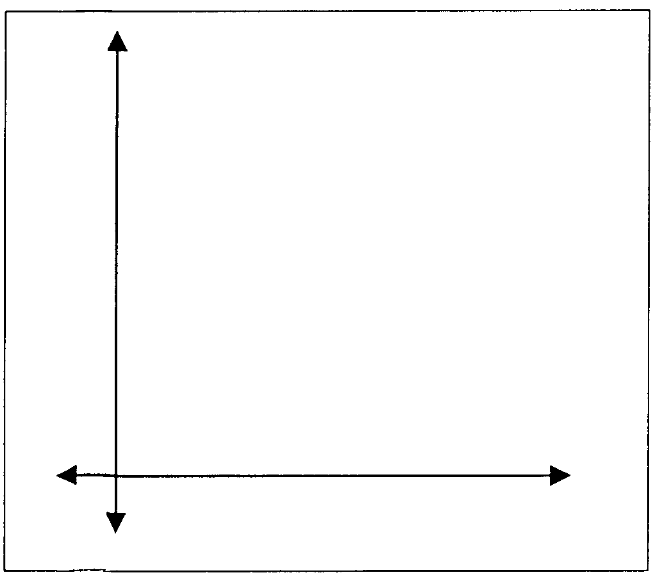
To find the regression equation using your TI-83, press STAT , move cursor to **CALC**. Select any model from **4: LinReg(ax + b)** up to **C:SinReg** and push enter. (If you like, do this for each type of equation.) Compare the regression coefficients to see which equation provides the best fitting curve.

Equation____

13. Tiger is positive that he can hit the ball, if the horizontal distance between the ball and its image is approximately 1 inch. Based on your equation is it possible for Tiger to hit the ball? Explain.

14. Are you satisfied with your equation? Do you think you could have found a better fitting model? Explain.

15. How would your graph change if you kept the height above water constant and the depth of the water was your independent variable?



Teacher Notes

The experiment can be performed as a “whole” group activity where you and a couple of students demonstrate the experiment and everyone uses the same data or in small groups.

If you decide to do the experiment in small groups:

- Set up the equipment for each group in advance.
- Have no more than four students in each group.

  One Observer – This person slides the index card to a given height using the line on the index card (and a little math) to determine the height above water level. He/she then looks through the peek hole and sights the image of the object. To be consistent the observer should place his or her eye close to the hole, so that the tip of his or her nose touches the index card.

  Two Readers – One reader will have two jobs: the first job is to slide the ruler (or wire) along the top of the container and away from the observer. The observer will tell the reader to stop when the ruler (or wire) appears to be on top and in the center of the small object that is submerged in the water. His/her second job is to read the tape measure on the right side of the container indicating the position of the ruler (or wire) for each trial. The second reader will read the tape measure on the left side of the container indicating the position of the ruler (or wire) for each trial.

  The ruler (or wire) <u>must</u> be kept perpendicular to the sides of the container to guarantee that the measures are accurate. To ensure that the ruler is perpendicular to the sides of the container, both readers should arrive at the same measure for each trial.

  One Recorder - This person records the height above water, and the distance of the image for each trial. He/she then calculates the average or median for the 3 trials and determines the horizontal distance between the object and image using the average (or median). This procedure is to be repeated for at least 12 different heights.

- Be sure that each student understands his/her role, what is to be measured and that each measurement is made as accurately as possible.
- Have each group perform three trials at the same height. Determine whether the recorder will be using the average or median of the three trials to determine the horizontal distance between the object and image.

Answers to LAB WRITE UP:

1. The purpose of the experiment is to determine the distance that Pooch and Tiger should drop the arrow to win the Big Prize.

2. a. Graph will vary. Assume correct if the graph is a decreasing function.
The graph is a decreasing function because as the height above water level increase the horizontal distance between the object and its image decreases.
b. By subtracting the horizontal distance of the image from the horizontal distance of the object you obtain the horizontal distance between the object and the image.

**3. Depth of water: 8 inches; Object Distance: 12 inches;
Height above water level: 2 inches Image Distance: 3 inches

Height Above Water	Distance of image	Hor. Dist. Between object & Image
2	3	9
3	3.75	8.25
4	4.63	7.37
5	5.25	6.75
6	5.63	6.37
7	6.13	5.87

Height Above Water	Distance of image	Hor. Dist. Between object & Image
8	6.5	5.5
9	6.88	5.12
10	7	5
11	7.13	4.87
12	7.25	4.75
13	7.5	4.5

Height Above Water	Distance of image	Hor. Dist. Between object & Image
14	7.7	4.3
15	7.78	4.22
16	7.9	4.1
17	7.95	4.05
18	8	4

4. a. The independent variable represents the Height Above Water Level.
The dependent variable represents the Horizontal Distance between the object and its Image.

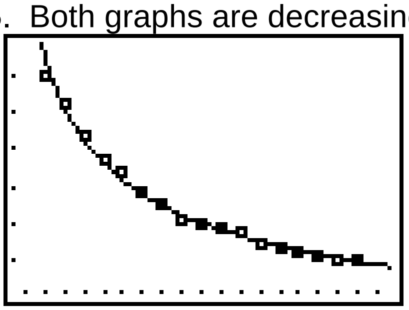
**b. Domain:

2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

Range:

9	8.25	7.37	6.75	6.37	5.87	5.5	5.12	5	4.87	4.75	4.5	4.3	4.22	4.1	4.05	4

5. Both graphs are decreasing functions.



6. a. The graph is nonlinear and hyperbolic in shape.

b. The taller person or greater the height above water level, the closer the image is to the object.

c. Yes, the height above water and the distance between the object and its image can have any value between the *maximum* and *minimum* domain/range values. Height and distance are continuous variables.

d. It makes sense to extend the curve to the left of the first data point up to 0 (the height above water level) at 0 your line of sight is on the water level and thus no image is produced. It makes sense to extend the curve beyond the right data point only to the distance of the object, because the distance of the image cannot be larger than the distance of the object.

e. Domain: Height above water level > 0
Range: 0 < Distance between the object and its image < the object distance.

f. No. The x-intercept represents the point where horizontal distance between the object and image is 0. Experimentally there is a point in which the difference is so small that it is insignificant. However theoretically, the difference is only zero when the object distance is zero, in other words the person is standing directly above the object.

g. No. The y-intercept represents the point where height above water level is zero. When the height above water level is zero no image is produced.

7. A hyperbolic function.

8. Answers will vary.

9. Answers will vary.

10. The greater the height above water level, the closer the image is to the object.
**Based on the sample (the graph), Pooch will hit the ball if he is at least 18 inches above water level and drops the arrow approximately 3 inches beyond the point where he sights the image of the ball. The graph seems to level off where the difference between the object and image is approximately 3 inches.

**11. If Pooch's eye level is 10.5 inches above water, then the corresponding distance between the object and its image is approximately 5 inches. Since the object is 5 inches away from the image, the object will be located at a point 7 + 5 or 12 inches away.

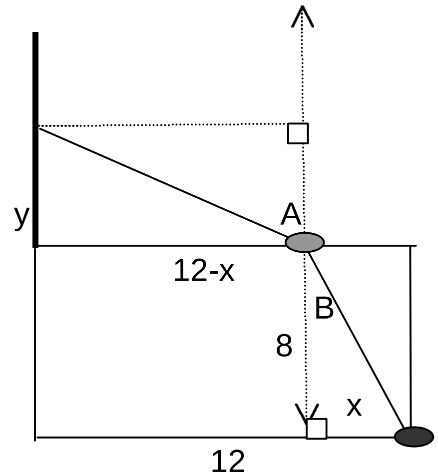
**12. The hyperbola
Using the calculator the power regression fits the data *best* – equation: $y = 12.5x^{-0.4}$

**13. If the distance between the object and its image is 1 inch. Then $12.5x^{-0.4} = 1$,

$$12.5x^{-0.4} = 1$$
$$x^{-0.4} = .08$$
$$x = .08^{-1/0.4}$$
$$x = .08^{-2.5}$$
$$x = 552.4$$

Tiger's eye level must be 552.4 inches or 46 feet above water level. It is not practical to obtain a height so large and still be able to spot the image.

14. Answers will vary.
    To get an equation algebraically:



-Draw a diagram of the problem situation.

-Use Snell's Law:

$$\frac{\sin A}{\sin B} = n$$

where A is the angle of incidence,
B is the angle of refraction and
n is the index of refraction.

the index of refraction for water is $\frac{4}{3}$

** $$\frac{\sin A}{\sin B} = \frac{4}{3}$$

$$3 \sin A = 4 \sin B$$

$$3\frac{12 - x}{\sqrt{(12 - x)^2 + y^2}} = 4\frac{x}{\sqrt{x^2+8^2}}$$

$$3(12 - x)\sqrt{x^2 + 8^2} = 4x\sqrt{(12 - x)^2 + y^2}$$

$$[3(12 - x)\sqrt{x^2 + 8^2}]^2 = [4x\sqrt{(12 - x)^2 + y^2}]^2$$

$$9(12 - x)^2(x^2 + 8^2) = 16x^2[(12 - x)^2 + y^2]$$

$$y^2 = \frac{9(12-x)^2(x^2 + 8^2) - 16x^2(12 - x)^2}{16x^2}$$

$$y^2 = \frac{(12 - x)^2}{16x^2}[9(x^2 + 8^2) + 16x^2]$$

$$y = \frac{12 - x}{4x}\sqrt{9(8)^2 - 7x^2}$$

$$f(x) = \frac{12 - x}{4x}\sqrt{576 - 7x^2}$$

The height above water vs. distance between the image and object can be modeled by the inverse of f(x).

15. As the depth of the water increases, the distance between the object and image increases.

**The answer is based on the sample. Your students' answer will vary.

Rubric

4 Demonstrates mastery over the process of making consistent measurements, collecting and organizing data, and constructing a mathematical model.

Provides accurate interpretations that illustrate insight into the information derived from the data.

3 Carries out the process of collecting and organizing data, and constructing a mathematical model without significant error.

Provides interpretations that are valid and appropriate for the data.

2 Makes numerous measurement and/or procedural errors such as making measurements that are inconsistent with each other.

Provides interpretations that are based on significant misunderstandings of the subject matter.

1 Makes numerous critical measurement and/or procedural errors.

Makes interpretations that have no clear bearing on the subject area or are clearly illogical.

Extension/Follow Up:

Use data and triangle trigonometry to calculate the index of refraction of water.

As light travels from one medium to another it changes the speed at which it propagates. For instance, the speed of light through air is greater than the speed of light through water or glass. The effect of this is the light changes direction or bends from a straight-line path at the boundary of the two media. We see this as a shift in an object's position.

We measure the relative bending of light using the concept of *index of refraction*. When two media have indexes of refraction that are nearly the same, there is very little bending at their boundary. When the difference of the indexes of refraction is large, we see a large amount of bending.

The index of refraction is easy to calculate using geometry and triangle trigonometry and using *Snell's Law*. Snell's Law says that the sine of the angle of incident divided by the sine of the angle of refraction equals the ratio of the refractive index of the second medium divided by the refractive index of the first medium. When the first medium is air, the expression can be simplified because the refractive index of air is very close to 1. Thus, we have the following equation where A is the angle of incident, B is the angle of refraction, and n is the refractive index of the second medium.

$$\frac{\sin A}{\sin B} = n$$

Consider the diagram on the following page. The index of refraction is calculated using Snell's Law by first taking careful measurements of the legs of the triangles (a, b, c, and d) and using triangle trigonometry to compute the sines of the two angles. Thus:

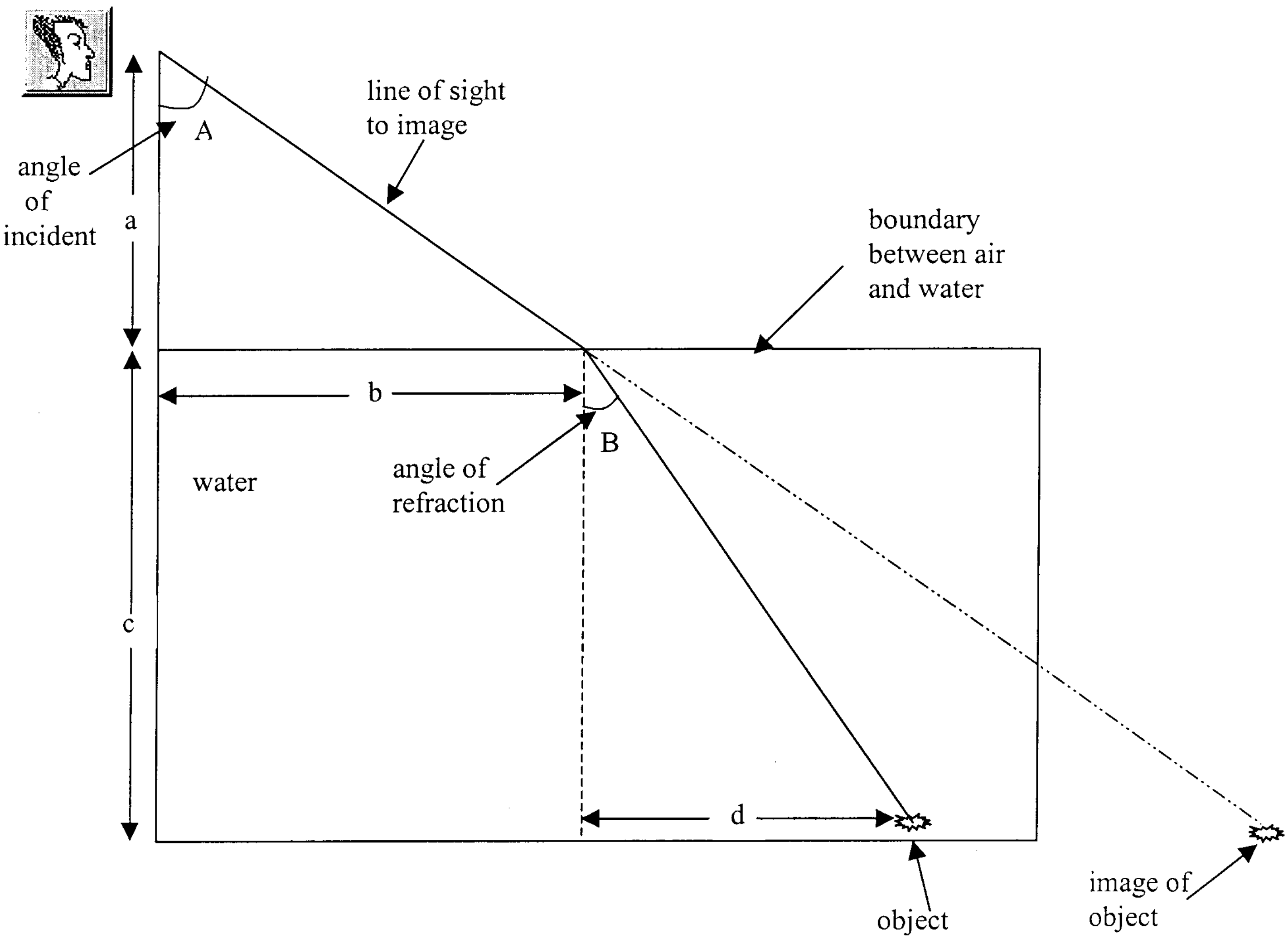
$$\sin A = \frac{opposite}{hypotenuse} = \frac{b}{\sqrt{a^2 + b^2}}$$

$$\sin B = \frac{opposite}{hypotenuse} = \frac{d}{\sqrt{c^2 + d^2}}$$

$$\text{Index of Refraction for water} = \frac{\sin A}{\sin B} = n$$

Compare your calculations with the actual refractive index for water, which is 1.3336. The indexes for other transparent materials are vacuum 1.0000, air (at room temperature and sea level) 1.000292, water 1.3336, glass (depending on its composition) typically 1.5 to 1.65, and diamond 2.42.

The relative refractive property of different transparent materials and the resulting bending of the light at the boundary are the basic principle behind optical lens. An additional extension of this exercise is calculating the proportional speed of light in different media. It turns out that the time that light travels from the boundary point to the object is the same as the time it travels from the boundary point to the image. By comparing the respective distances, a student can see that the speed of light to the (closer) object is less than speed of light to the (further) image.



Assessment:

Select the BEST answer:

1. If two people are standing the same distance from a target that is underwater, the person viewing the target from a higher position will:

   a. not see the target as well because his or her viewing angle is closer to perpendicular.
   b. not see the target as well because they are further from it
   c. see the target closer to its true position because there is less bending of the light.
   d. need to move directly over the target to make sure that the target is seen at its true position.

For problems 2 through 4. Refer to an experiment in which Pooch is standing approximately 15 inches from the tank. He took several measurements of the height above water and the corresponding distances between the object and its image.

Height Above Water	Distance of image	Hor. Dist. Between object & Image
2	3	9
3	3.75	8.25
4	4.63	7.37
5	5.25	6.75
6	5.63	6.37
7	6.13	5.87

Height Above Water	Distance of image	Hor. Dist. Between object & Image
8	6.5	5.5
9	6.88	5.12
10	7	5
11	7.13	4.87
12	7.25	4.75
13	7.5	4.5

Height Above Water	Distance of image	Hor. Dist. Between object & Image
14	7.7	4.3
15	7.78	4.22
16	7.9	4.1
17	7.95	4.05
18	8	4

2. Make a scatter plot for the data on the grid provided. Title your scatter plot and label the horizontal and vertical axes.

3. Using the scatter plot above and the problem situation, would you describe the relationship between the variables as

   a. linear
   b. nonlinear and hyperbolic
   c. nonlinear and exponential
   d. nonlinear and parabolic

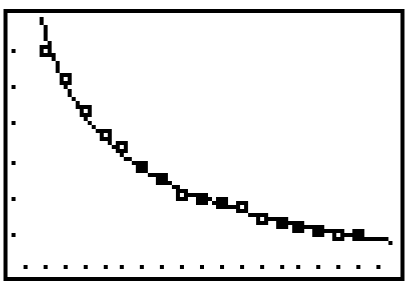
4. Using your graphing calculator, find the equation of the best fit line or curve (using the appropriate regression).

5. Using the table, graph the problem situation above. What would you expect the data to do if it were extended further on either end; that is, if you view the object from a point lower and higher than the points given in the data table?

Answer Key:

1. c

2.



3. b

4. equation: $y = 12.5x^{-0.4}$

5. Below 2 you will get large changes in the horizontal distances, above 18 the changes in the horizontal distances are small.